Malaysian Journal of Mathematical Sciences 19(3): 1111–1131 (2025)
https://doi.org/10.47836/mjms.19.3.17

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Investigation of Novel Symmetry Solutions and Conservation Laws of a Generalized Double Dispersion Equation in Uniform and Inhomogeneous Murnaghan's Rod

Monashane, M. S.[1], Adeyemo, O. D.[1], and Khalique, C. M.*[1,2]

[1]*Material Science, Innovation and Modelling Focus Area,*
Department of Mathematical Sciences, North-West University,
Mafikeng Campus, Private Bag X2046, Mmabatho 2735, Republic of South Africa
[2]*Department of Mathematics and Informatics, Azerbaijan University,*
Jeyhun Hajibeyli str., 71, AZ1007, Baku, Azerbaijan

E-mail: Masood.Khalique@nwu.ac.za
**Corresponding author*

Received: 29 July 2024
Accepted: 15 April 2025

Abstract

Dispersion is a phenomenon in which a wave's phase velocity varies with its frequency. It can be relevant to all kinds of wave movements, including sound, seismic waves, and gravitational waves. The double dispersion equation is important due to its numerous physical applications, such as examining the nonlinear wave distribution in waveguides, investigating the interaction of waveguides with the surrounding medium, and assessing the probability of energy transfer through lateral waveguide coverings. In view of this, this article explores analytical examinations of a (1+1)-dimensional generalized double dispersion equation in inhomogeneous and uniform Murnaghan's rod. This is applicable in modeling wave propagation in an elastic solid material, which has significance in solid-state mechanics. Therefore, it is entrenched in solid-state physics. Lie group theory is invoked to identify point symmetries associated with the model, enabling the derivation of nonlinear ordinary differential equations through symmetry reduction. Furthermore, direct integration of the nonlinear ordinary differential equation is performed to obtain closed-form solutions to the underlying model. Consequently, an elliptic cosine function solution is attained. Additionally, using a specific transformation, the technique further ensures the attainment of a Weierstrass function solution. To secure more solutions to the studied equation, the well-known Kudryashov's method is utilized, affording us the opportunity to obtain an exponential function solution. Subsequently, we applied the (G'/G)-expansion technique, which consequently produces hyperbolic, rational, and trigonometric function solutions. Moreover, to view the wave dynamics of the achieved solutions, which provides us with the opportunity to capture the physical meanings of these solutions, various wave depictions are demonstrated in three-dimensional, two-dimensional, contour, and density plots. In conclusion, the study produces notable conserved quantities such as energy, mass, and momentum, which are secured using Ibragimov's theorem, as well as the multiplier approach.

Keywords: generalized double dispersion equation; Lie symmetry method; closed form solutions; Kudryashov's and (G'/G)-expansion techniques; conservation laws.

1 Introduction

In the world around us, we encounter a multitude of complex physical phenomena that exhibit nonlinearity. These phenomena are accurately described by Nonlinear Partial Differential Equations (NPDEs), with examples ranging from population ecology and epidemiology to biology, plasma physics, fluid mechanics, and nonlinear circuits. Nonlinear partial differential equations are significant as they represent numerous real-world issues across various disciplines, such as in Physics, NPDEs are used to represent physical problems like fluid dynamics, solid mechanics, acoustics, plasma physics, and quantum field theory. In life sciences, NPDEs are utilized in chemical and biological systems, as well as for modeling population biology, predator-prey interactions, and other scenarios. Additionally, in engineering, NPDEs are employed to address practical issues. In mathematics, NPDEs have been used to tackle mathematical problems like the Poincaré conjecture and the Calabi conjecture. NPDEs pose challenges for study as there are limited universal methods applicable to all equations. Typically, each equation must be examined separately [1-20].

To gain a deep understanding of these phenomena, it is essential to investigate the solutions to the differential equations (DEQNs) that govern them. Thus, this necessitates investigation of solitary wave solutions of these NPDEs in their exact structure. Extensive research continues to be conducted on these equations, as they play a pivotal role in modeling relationships between various physical quantities found in the natural world. Recent advancements in computer technology have greatly improved our ability to develop algorithms for solving NPDEs. Despite this progress, it is important to acknowledge the brilliant minds that have laid the theoretical groundwork for these technologies to flourish. In recent times, numerous researchers with a strong interest in nonlinear physical phenomena have explored closed-form solutions of NPDEs due to their significance in analyzing model outcomes. It is vital that research on closed-form solutions to NPDEs plays a crucial role in understanding specific physical scenarios. The range of solutions to NPDEs holds a significant position in various scientific fields. These include electromagnetic theory, chemical physics, optical fibers, hydrodynamics, meteorology, plasma physics, biology, heat flow, chemical kinetics, and geochemistry.

Recognizing that many prominent scientists view nonlinear science as a key frontier for gaining a deeper understanding of nature, we present a few relevant models, including a 3D generalized nonlinear potential model, the YTSF equation in Engineering and Physics, recently examined in [2]. Furthermore, an examination in another source focused on the modified and generalized ZKe model, highlighting ion-acoustic solitary waves found in a magnetoplasma environment containing electron-positron-ion particles present in a native universe [11]. The authors in the reference delve into applications of the model in studying waves in dust-ion acoustics, dust-magneto, and ions within laboratory dusty plasmas.

Moreover, studies in [40] explored bright solitons (vectors) and their interactions within a coupled Fokas-Lenells model. The investigation also extended to optical pulses' femtosecond embedded in double-refractive fibers in optics, modeled using NPDEs. Additionally, attention was given to a type of Boussinesq-Burgers model system recounting waves embedded in shallow water near ocean shores and lakes, as discussed in [13]. The text also references other related studies for further exploration. The listed publication in [3] can be visited also in which a type of three dimensional generalized Zakharov-Kuznetsov equation was investigated. Additionally, the application of topological and non-topological solitons in the fields of physical and nonlinear sciences were highlighted.

It is well known that there is no universal approach for achieving exact solutions to NPDEs.

However, to address this persistent issue, researchers have developed several effective techniques. For instance, Sophus Lie (1842–1899) made significant contributions to the field with his investigations on Lie algebras [29, 30], providing a unified technique for solving a vast range of DEQNs. Other notable study using Lie symmetry approach can be found in Kumar and Dhiman [24] where they invoked Lie symmetry approach and the unified technique to solve a coupled breaking soliton model. This offers a universal and robust method by which exact solutions to differential equations can be obtained systematically. However, the limitation of the approach lies in the fact that it can only be applied to differential equations that have symmetries.

Recent advancements in solving DEQNs include Kudryashov's approach [21], the simplest equation technique [34], the sine-Gordon equation expansion technique [6], Hirota's bilinear approach [26], the (G'/G)-expansion technique [35], the power series solution technique [12], the Darboux transformation approach [42]. Additional techniques such as the Painlevé expansion technique [8], the bifurcation approach [41], homotopy perturbation [7], extended homoclinic test technique [9], tanh-coth technique [38], Adomian decomposition technique [37], symmetry group analysis [29, 30], F-expansion approach [43], Bäcklund transformation method [15], extended simplest equation technique [21], Cole-Hopf transformation approach [31], rational expansion approach [39], and many more have been developed.

Since the establishment of Petviashvili and Kadomtsev's hierarchy equation models over 50 ago, numerous research papers have been published, each exploring different aspects of this complex field of equations. For instance, in the publication given in [25], Kuo and Ma proposed an effective algorithm for constructing nonlinear evolution equations. In the same vein, Madhavan et al. [27] examined a pursuit differential game involving one pursuer and one evader for a higher-level infinite system comprised of first-order ternary differential equations, and demonstrated the successful completion of the pursuit within the game. Besides, the authors in [33] investigated the issue that the Korteweg-de Vries equation fails to completely represent the intricacy of nonlinear waves. To tackle this challenge, they resolved the extended Korteweg-de Vries equation, incorporating higher-order nonlinear and dispersion components. The primary aim was to explore the influence of cubic nonlinearity and fifth-order dispersion on solitary wave propagation.

One of the interesting and highly applicable fourth-order NPDEs is a double dispersion equation (DoDEqn) in an inhomogeneous and uniform Murnaghan rod, explicated as [32],

$$\begin{aligned}\frac{\partial^2 w(x,t)}{\partial t^2} - \left(\frac{E}{\rho}\right)\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\beta}{\rho}\left\{w(x,t)\frac{\partial^2 w(x,t)}{\partial x^2} + \left(\frac{\partial w(x,t)}{\partial x}\right)^2\right\}& \\ - \left(\frac{\nu^2 R_0^2}{2}\right)\frac{\partial^4 w(x,t)}{\partial t^2 \partial x^2} + \left(\frac{\nu^2 R_0^2 \sigma}{2\rho}\right)\frac{\partial^4 w(x,t)}{\partial x^4} = 0,& \qquad (1)\end{aligned}$$

where E represents Young's modulus, β stands for the nonlinear coefficient, σ connotes the Lamé coefficient, ν represents the Poisson ratio while n_1, l, ρ, n_2, R_0, and ρ are constants. Dispersion is a phenomenon in which a wave's phase velocity varies with its frequency. It can be relevant to all kinds of wave movements, including sound, seismic waves, and gravitational waves. In optics, dispersion refers to a characteristic of light and other electromagnetic waves. The DoDEqn model is important due to its numerous physical applications, such as examining the nonlinear wave distribution in waveguides, investigating the interaction of waveguides with the surrounding medium, and assessing the probability of energy transfer through lateral waveguide coverings. This equation is applicable in modeling wave propagation in elastic solid materials, which has significance in solid-state mechanics.

In his work, Samsonov [32] developed a range of double dispersive equations for solid materials containing elastic waveguides and rods with complex properties. The Euler equation is used to derive the four forms of the double dispersive equations and they are:

(a) Movement within the heterogeneous and non-uniform rod.

(b) Displacement within the heterogeneous and uniform rod.

(c) Movement within the homogeneous and non-uniform rod.

(d) Movement within the homogeneous and uniform rod.

Additionally, elliptic functions, solitary waves, as well as numerical solutions were also entrenched by the author.

The DoDEqn (1), also referred to as Euler's displacement model in heterogeneous and uniform Murnaghan's rod [32], has been investigated by a few other researchers. Nisar and Silambarasan [28] utilized the F-expansion technique on the double dispersion model in Murnaghan's rod to attain a Jacobi elliptic function solution and categorized it into six distinct solution families. Each solution is accompanied by the necessary condition, and the degeneration of the Jacobi solutions is based on the modulus of the elliptic function. The solutions derived from algebraic equations determine the formation of the six classifications. Additionally, Cattani et al. [5] were able to solve the double dispersive model in the non-uniform circular cylindrical rod for both bright and dark solitons by invoking the modified $\exp\left[-\varphi\left(\zeta\right)\right]$ function alongside extended sinh-Gordon techniques.

Having explored the various work done on model (1), there is a gap that still needs to be filled, which has led us to consider the study of a generalized version of the equation (significant in solid - state physics) from a Lie group analysis perspective. Therefore, the (1+1)-dimensional Generalized Double Dispersion Equation ((1+1)D-GnDDE) to be investigated in this work is presented as,

$$\Delta \equiv w_{tt} + aw_{xx} + b(ww_{xx} + w_x^2) + cw_{ttxx} + dw_{xxxx} = 0, \tag{2}$$

in which constant coefficients a to d are all nonzero real valued. We declare here that, to the best of our knowledge, the generalized model (2) has not been investigated previously. To carry out this study, the article is structured as follows: The introduction and literature review of the work done on the model under investigation are presented in Section 1. Section 2 contains the procedural pattern through which Lie point symmetries of (1+1)D-gnDDE (2) are obtained. Additionally, various approaches are explored to secure analytic solutions of interest to (2). Moreover, in Section 3, conservation laws associated with the model under study are calculated using Ibragimov's theorem along with the multiplier approach. Thereafter, the concluding remarks follow in Section 4.

2 Symmery Analysis and Solutions of (2)

We begin by deriving the Lie point symmetries of (1+1)D-gDDE (2), after which we use them to obtain its exact solutions.

2.1 Lie point symmetries of (2)

The symmetry group of (1+1)D-gnDDE (2) will be formed by the vector field,

$$Q = \xi^1(x,t,w)\frac{\partial}{\partial t} + \xi^2(x,t,w)\frac{\partial}{\partial x} + \varphi(x,t,w)\frac{\partial}{\partial w},$$

with coefficients ξ^i, $i = 1, 2$ and φ being functions of (x, t, w), is a Lie point symmetry of (1+1)D-gnDDE (2) if,

$$Q^{[4]}\left[w_{tt} + aw_{xx} + b(ww_{xx} + w_x^2) + cw_{ttxx} + dw_{xxxx}\right]\Big|_{\Delta=0} = 0, \tag{3}$$

where $Q^{[4]}$ represents the fourth extension of vector field Q, which is defined as,

$$Q^{[4]} = Q + \zeta^t\frac{\partial}{\partial w_t} + \zeta^x\frac{\partial}{\partial w_x} + \zeta^{tt}\frac{\partial}{\partial w_{tt}} + \zeta^{xx}\frac{\partial}{\partial w_{xx}} + \zeta^{ttxx}\frac{\partial}{\partial w_{ttxx}} + \zeta^{xxxx}\frac{\partial}{\partial w_{xxxx}}, \tag{4}$$

with the ζ's given as:

$$\begin{aligned}
\zeta^t &= D_t(\eta) - w_t D_t(\xi^1) - w_x D_t(\xi^2),\\
\zeta^x &= D_x(\eta) - w_t D_x(\xi^1) - w_x D_x(\xi^2),\\
\zeta^{tt} &= D_t(\zeta^t) - w_{tt} D_t(\xi^1) - w_{tx} D_t(\xi^2),\\
\zeta^{xx} &= D_x(\zeta^x) - w_{tx} D_x(\xi^1) - w_{xx} D_x(\xi^2),\\
\zeta^{ttxx} &= D_x(\zeta^{ttx}) - w_{tttx} D_x(\xi^1) - w_{ttxx} D_x(\xi^2),\\
\zeta^{xxxx} &= D_x(\zeta^{xxx}) - w_{txxx} D_x(\xi^1) - w_{xxxx} D_x(\xi^2),
\end{aligned}$$

and the total derivatives are given as:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + w_t\frac{\partial}{\partial w} + w_{tt}\frac{\partial}{\partial w_t} + w_{tx}\frac{\partial}{\partial w_x} + w_{txx}\frac{\partial}{\partial w_{xx}} + w_{tttxx}\frac{\partial}{\partial w_{ttxx}} + \ldots,\\
D_x &= \frac{\partial}{\partial x} + w_x\frac{\partial}{\partial w} + w_{xx}\frac{\partial}{\partial w_x} + w_{tx}\frac{\partial}{\partial w_t} + w_{ttxxx}\frac{\partial}{\partial w_{ttxx}} + w_{xxxxx}\frac{\partial}{\partial w_{xxxx}} + \ldots.
\end{aligned} \tag{5}$$

By expanding (3) and separating it based on the derivatives of the function w, we can derive the following system of overdetermined linear partial differential equations (LPDEQs):

$$\xi^1_t = 0, \quad \xi^1_w = 0, \quad \xi^1_x = 0, \quad \xi^2_t = 0, \quad \xi^2_w = 0, \quad \xi^2_x = 0, \quad \varphi = 0,$$

which can easily be solved and so, the solution to the system yields,

$$\xi^1(x,t,w) = B_1 \quad \text{and} \quad \xi^2(x,t,w) = B_2,$$

in which B_1, together with B_2 are arbitrary constants. Thus, the Lie point symmetries of (1+1)D-gnDDE (2) are translational, given as,

$$Q_1 = \frac{\partial}{\partial t} \quad \text{and} \quad Q_2 = \frac{\partial}{\partial x}. \tag{6}$$

2.2 Travelling wave solutions of (1+1)D-gnDDE (2)

On considering a combination of the obtained translational symmetries Q_1 and Q_2 for (1+1)D-gnDDE (2) as $Q = Q_1 + \omega Q_2$. The symmetry produces the two invariants,

$$\zeta = x - \omega t, \quad U = w,$$

that yield $U = U(\zeta)$ (wherein ζ is the new independent variable) as the group-invariant. One makes use of the above and successfully transforms model (2) to the nonlinear ordinary differential equation (NLNODE) of fourth-order, viz.,

$$aU''(\zeta) + b\left\{U(\zeta)U''(\zeta) + U'^2(\zeta)\right\} + c\omega^2 U''''(\zeta) + dU''''(\zeta) + \omega^2 U''(\zeta) = 0. \tag{7}$$

2.2.1 Exact solutions of (2) through direct integration

The direct integration technique to be invoked here will furnish three categories of solutions to (1+1)D-gnDDE (2). These will take the structure of Jacobi elliptic, Weierstrass as well as hyperbolic secant functions.

- **Jacobi elliptic cosine function solution**

  One commences by integrating (7) twice with regard to independent variable ζ and the result purveys,

$$aU(\zeta) + \frac{1}{2}bU(\zeta)^2 + c\omega^2 U''(\zeta) + dU''(\zeta) + \omega^2 U(\zeta) + A_0\zeta + A_1 = 0, \tag{8}$$

  wherein integration constants $A_m, m = 0, 1$, are arbitrary. One takes A_0 as zero and integrate the rest of the equation after multiplying it by $U'(\zeta)$. The outcome yields,

$$\left(\frac{a}{c\omega^2} + \frac{1}{c}\right)U(\zeta)^2 + \frac{2A_1}{c\omega^2}U(\zeta) + \frac{b}{3c\omega^2}U(\zeta)^3 + \left(\frac{d}{c\omega^2} + 1\right)U'^2(\zeta) + \frac{2A_2}{c\omega^2} = 0. \tag{9}$$

  In consequence, we achieve,

$$U'^2(\zeta) = -\left\{PU(\zeta)^3 + QU(\zeta)^2 + RU(\zeta) + S\right\}, \tag{10}$$

  where

$$P = \frac{b}{3\left(c\omega^2 + d\right)}, \quad Q = \left(\frac{\omega^2}{c\omega^2 + d} + \frac{a}{c\omega^2 + d}\right), \quad R = \frac{2A_1}{c\omega^2 + d} \quad \text{and} \quad S = \frac{2A_2}{c\omega^2 + d}. \tag{11}$$

  Suppose one contemplates the cubic function,

$$U(\zeta)^3 + \frac{Q}{P}U(\zeta)^2 + \frac{R}{P}U(\zeta) + \frac{S}{P} = 0, \tag{12}$$

  whose roots are α_0, α_1 and α_2 in such a way that $\alpha_0 > \alpha_1 > \alpha_2$. Therefore, NLNODE (10) explicates as,

$$U'^2(\zeta) = -P\left\{(U - \alpha_0)(U - \alpha_1)(U - \alpha_2)\right\}. \tag{13}$$

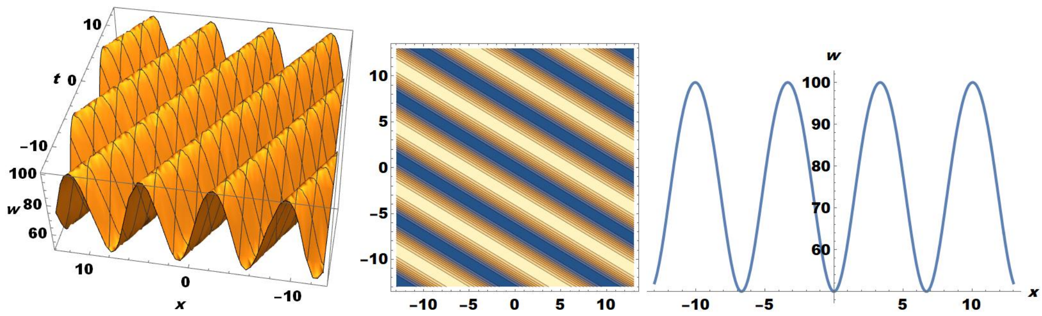
  Hence, (13) has the solution [20, 1],

$$U(\zeta) = \alpha_1 + (\alpha_0 - \alpha_1)\,\mathrm{cn}^2\left\{\sqrt{\frac{P(\alpha_0 - \alpha_2)}{4}}(\zeta - \zeta_0)\middle| M^2\right\}, \ M^2 = \frac{\alpha_0 - \alpha_1}{\alpha_0 - \alpha_2}, \tag{14}$$

  where Jacobi cosine function cn as well as constant ζ_0 exist. Returning to the original variables gives the periodic solution of (1+1)D-gnDDE (2) as,

$$w(x,t) = \alpha_1 + (\alpha_0 - \alpha_1)\,\mathrm{cn}^2\left\{\sqrt{\frac{b(\alpha_0 - \alpha_2)}{12\left(c\omega^2 + d\right)}}\,(x - \omega t - \zeta_0)\middle| M^2\right\}. \tag{15}$$

The wave dynamics of periodic solution (15) is exhibited in the plots explicated in Figure 1.



(a) (b) (c)

Figure 1: The smooth periodic wave structure is demonstrated through the plots in 3D, 2D, and contour formats depicting the elliptic solution (15) and is displayed using the parameter assignments: $b = 0.2$, $c = 0.1$, $\alpha = 2$, $\beta = -1.2$, $d = 0.4$, $\omega = 20$, $\alpha_0 = 100$, $\alpha_1 = 50.05$, and $\alpha_2 = -60$ in the interval $-13 \leq t, x \leq 13$.

- **Weierstrass function solution**

Here, one retrieves a periodic solution to NLNODE (7) in terms of Weierstrass function [23] by setting in (9), a transformation as,

$$U(\zeta) = -\frac{1}{b}\left\{\left(\omega^2 + a\right) + 12\left(c\omega^2 + d\right) W(\zeta)\right\}. \tag{16}$$

Hence, one reckons (9) as NLNODE with Weierstrass elliptic function [14, 4],

$$\wp'(\zeta)^2 - 4\wp(\zeta)^3 + g_1\wp(\zeta) + g_2 = 0, \tag{17}$$

with the included Weierstrass elliptic invariants g_1 as well as g_2 expressed as,

$$g_1 = \frac{1}{b}\left(24A_1 - \frac{12a^2}{b} - \frac{24a\omega^2}{b} - \frac{12\omega^4}{b}\right),$$
$$g_2 = \frac{1}{b}\left(24A_2 + \frac{8a^3}{b^3} + \frac{24a^2\omega^2}{b^3} - \frac{24aA_1}{b^2} + \frac{24a\omega^4}{b^3} - \frac{24A_1\omega^2}{b^2} + \frac{8\omega^6}{b^3}\right).$$

Hence, solution to NLNODE (7) entrenches in this regard,

$$U(\zeta) = \wp\left\{\frac{1}{2}\zeta\sqrt{\left|-\frac{b}{3\left(c\omega^2+d\right)}\right|}; \frac{24A_1}{b} - \frac{12\omega^4}{b^2} - \frac{24a\omega^2}{b^2} - \frac{12a^2}{b^2}, \frac{8\omega^6}{b^3} + \frac{24a\omega^4}{b^3} - \frac{24A_1\omega^2}{b^2} + \frac{24a^2\omega^2}{b^3} - \frac{24aA_1}{b^2} + \frac{24A_2}{b} + \frac{8a^3}{b^3}\right\} - \frac{a+\omega^2}{b}. \tag{18}$$

Bearing in mind (16) alongside (17) and reverting to previous variables, one has,

$$w(x,t) = \wp\left\{\frac{1}{2}(x-\omega t)\sqrt{\left|-\frac{b}{3\left(c\omega^2+d\right)}\right|}; \frac{24A_1}{b} - \frac{12\omega^4}{b^2} - \frac{24a\omega^2}{b^2} - \frac{12a^2}{b^2}, \frac{8\omega^6}{b^3} + \frac{24a\omega^4}{b^3} - \frac{24A_1\omega^2}{b^2} + \frac{24a^2\omega^2}{b^3} - \frac{24aA_1}{b^2} + \frac{24A_2}{b} + \frac{8a^3}{b^3}\right\} - \frac{a+\omega^2}{b}, \tag{19}$$

where $\wp$ represents Weierstrass function [14].

- **Bright soliton solution**

  Here, one considers another case of NLNODE (9), whereby $A_1 = A_2 = 0$. This, leads to,

$$\left(\frac{a}{c\omega^2}+\frac{1}{c}\right)U(\zeta)^2+\frac{b}{3c\omega^2}U(\zeta)^3+\left(\frac{d}{c\omega^2}+1\right)U'^2(\zeta)=0. \tag{20}$$

  Solving the equation furnishes the soliton solution of model (2) as,

$$w(x,t)=\frac{1}{b}\left\{3\left(a+\omega^2\right)\operatorname{sech}^2\left[-\frac{1}{2}\left(B_0\sqrt{3\left(a+\omega^2\right)}+\sqrt{\frac{-\left(a+\omega^2\right)}{c\omega^2+d}}(x-\omega t)\right)\right]\right\}, \tag{21}$$

  where B_0 is an integration constant. Dynamics of the secant hyperbolic solution (21) is the plots explicated in Figure 2.

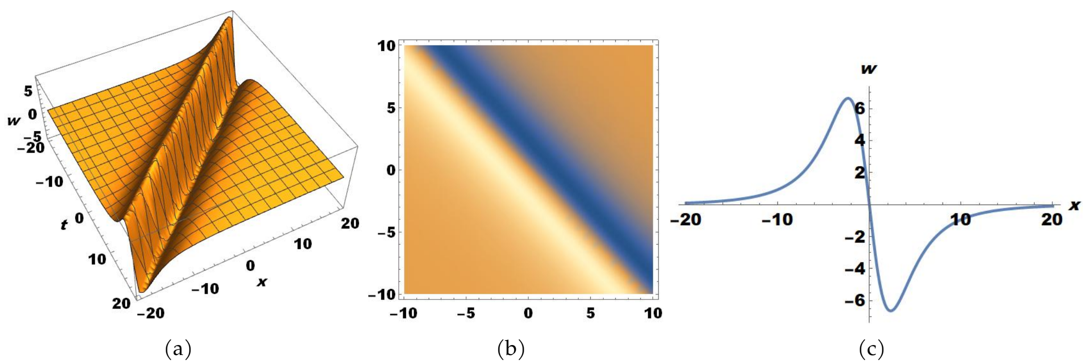


Figure 2: The combination of bright and dark soliton waves is exhibited through the bell-anti-bell-shaped wave structure of the hyperbolic secant function solution (21) using the data values $b = 2$, $c = 0.1$, $\alpha = 0.12$, $\beta = -0.13$, $d = 0.4$, $\omega = 0.01$, $a = -1$, and $B_0 = 5$ in the interval $-20 \leq t, x \leq 20$.

Now, in order to secure various more interesting closed - form solutions to (1+1)D-gnDDE (2), some standard approaches are used.

2.2.2 Exact solutions of (2) using Kudryashov's approach

In this subsection, we utilize Kudryashov's technique [22] to identify exact solutions for (2). Our starting point involves hypothesizing solutions for the fourth-order NLNODE (7) in a specific format,

$$U(\zeta)=\sum_{i=0}^{M}A_iH^i(\zeta), \tag{22}$$

in which $H(\zeta)$ fulfills Riccati equation,

$$H'(\zeta)=H^2(\zeta)-H(\zeta), \tag{23}$$

whose solution is given as,

$$H(\zeta)=\frac{1}{1+e^{\zeta}}. \tag{24}$$

Value of M in (22) can be achieved through the application of the balancing technique as referenced in [36]. The constants A_i for $i = 0, 1, \ldots, M$ will also be established through this process, thus producing $M = 2$.

One eventually, explicates (22) as,

$$U(\zeta) = A_0 + A_1 H(\zeta) + A_2 H(\zeta)^2. \tag{25}$$

By replacing assumption (25) into (7) and utilizing relation (23), we derive the subsequent expression in terms of $H(\zeta)$,

$$\begin{aligned}
&6\,aA_2\left(H\left(\zeta\right)\right)^4 + 2\,aA_1\left(H\left(\zeta\right)\right)^3 - 10\,aA_2\left(H\left(\zeta\right)\right)^3 + aA_1H\left(\zeta\right) - 3\,aA_1\left(H\left(\zeta\right)\right)^2\\
&+24\,dA_1\left(H\left(\zeta\right)\right)^5 + 4\,aA_2\left(H\left(\zeta\right)\right)^2 + 120\,dA_2\left(H\left(\zeta\right)\right)^6 - 60\,dA_1\left(H\left(\zeta\right)\right)^4\\
&+dA_1H\left(\zeta\right) + 50\,dA_1\left(H\left(\zeta\right)\right)^3 - 15\,dA_1\left(H\left(\zeta\right)\right)^2 - 336\,dA_2H\left(\zeta\right)^5 + 330\,dA_2H\left(\zeta\right)^4\\
&-130\,dA_2H\left(\zeta\right)^3 + 6\,\omega^2A_2\left(H\left(\zeta\right)\right)^4 + 16\,dA_2\left(H\left(\zeta\right)\right)^2 + \omega^2A_1H\left(\zeta\right) + 2\,\omega^2A_1H\left(\zeta\right)^3\\
&-3\,\omega^2A_1\left(H\left(\zeta\right)\right)^2 - 10\,\omega^2A_2\left(H\left(\zeta\right)\right)^3 + 4\,\omega^2A_2\left(H\left(\zeta\right)\right)^2 + 2\,bA_1^2\left(H\left(\zeta\right)\right)^2\\
&-5\,bA_1^2H\left(\zeta\right)^3 + 3\,bA_1^2\left(H\left(\zeta\right)\right)^4 + 10\,bA_2^2\left(H\left(\zeta\right)\right)^6 - 18\,bA_2^2H\left(\zeta\right)^5 + 8\,bA_2^2H\left(\zeta\right)^4\\
&-60\,\omega^2cA_1\left(H\left(\zeta\right)\right)^4 + 4\,bA_2\left(H\left(\zeta\right)\right)^2A_0 + 24\,\omega^2cA_1H\left(\zeta\right)^5 + 9\,bA_1\left(H\left(\zeta\right)\right)^3A_2\\
&+6\,bA_2\left(H\left(\zeta\right)\right)^4A_0 - 10\,bA_2\left(H\left(\zeta\right)\right)^3A_0 - 3\,bA_1\left(H\left(\zeta\right)\right)^2A_0 - 21\,bA_1H\left(\zeta\right)^4A_2\\
&+bA_1H\left(\zeta\right)A_0 - 130\,\omega^2cA_2\left(H\left(\zeta\right)\right)^3 + 16\,\omega^2cA_2\left(H\left(\zeta\right)\right)^2 + 2\,bA_1\left(H\left(\zeta\right)\right)^3A_0\\
&+12\,bA_1\left(H\left(\zeta\right)\right)^5A_2 + 50\,\omega^2cA_1\left(H\left(\zeta\right)\right)^3 - 15\,\omega^2cA_1\left(H\left(\zeta\right)\right)^2 + \omega^2cA_1H\left(\zeta\right)\\
&+120\,\omega^2cA_2\left(H\left(\zeta\right)\right)^6 - 336\,\omega^2cA_2\left(H\left(\zeta\right)\right)^5 + 330\,\omega^2cA_2\left(H\left(\zeta\right)\right)^4 = 0.
\end{aligned} \tag{26}$$

By comparing the coefficients of similar powers of $H(\zeta)$ in (26), one derives the subsequent five algebraic equations involving A_0, A_1, and A_2 that is:

$$\begin{aligned}
H^6(\zeta):&\quad 120\,\omega^2cA_2 + 10\,bA_2^2 + 120\,dA_2 = 0,\\
H^5(\zeta):&\quad 24\,\omega^2cA_1 - 336\,\omega^2cA_2 + 12\,bA_1A_2 - 18\,bA_2^2 + 24\,dA_1 - 336\,dA_2 = 0,\\
H^4(\zeta):&\quad 330\,\omega^2cA_2 - 60\,\omega^2cA_1 + 6\,bA_0A_2 + 3\,bA_1^2 - 21\,bA_1A_2 + 8\,bA_2^2\\
&\quad + 6\,\omega^2A_2 + 6\,aA_2 - 60\,dA_1 + 330\,dA_2 = 0,\\
H^3(\xi):&\quad 50\,c\omega^2A_1 - 130\,c\omega^2A_2 + 2\,bA_0A_1 - 10\,bA_0A_2 - 5\,bA_1^2 + 9\,bA_1A_2\\
&\quad + 2\,\omega^2A_1 - 10\,\omega^2A_2 + 2\,aA_1 - 10\,aA_2 + 50\,dA_1 - 130\,dA_2 = 0,\\
H^2(\zeta):&\quad 16\,\omega^2cA_2 - 15\,\omega^2cA_1 - 3\,bA_0A_1 + 4\,bA_0A_2 + 2\,bA_1^2 - 3\,\omega^2A_1\\
&\quad + 4\,\omega^2A_2 - 3\,aA_1 + 4\,aA_2 - 15\,dA_1 + 16\,dA_2 = 0,\\
H(\zeta):&\quad \omega^2cA_1 + bA_0A_1 + \omega^2A_1 + aA_1 + dA_1 = 0.
\end{aligned} \tag{27}$$

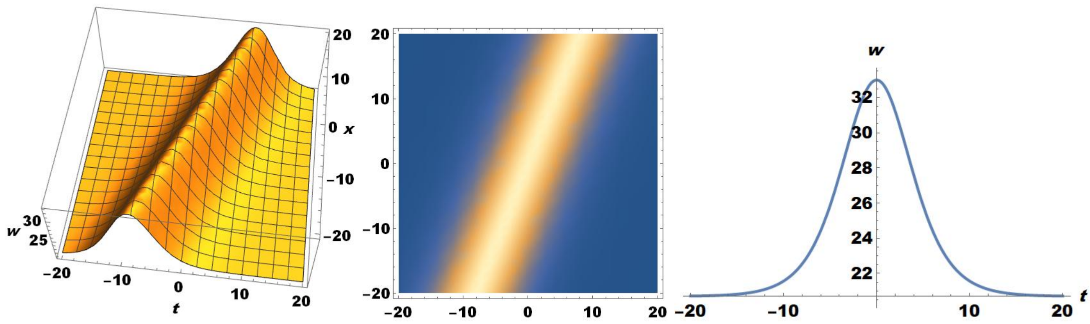
The solution to these equations thus produces,

$$A_0 = -\frac{1}{b}\left(c\omega^2 + \omega^2 + a + d\right), \quad A_1 = \frac{12}{b}\left(c\omega^2 + d\right), \quad A_2 = -\frac{12}{b}\left(c\omega^2 + d\right). \tag{28}$$

Thus, the solutions of (2) for the secured results respectively reads,

$$w(x,t) = -\frac{1}{b}\left\{\left(c\omega^2 + \omega^2 + a + d\right) - \frac{12\,\left(c\omega^2 + d\right)}{1 + e^{(x-\omega t)}} + \frac{12\,\left(c\omega^2 + d\right)}{\left(1 + e^{(x-\omega t)}\right)^2}\right\}. \tag{29}$$

The wave structure of the exponential solution (29) is plotted in Figure 3.



(a) (b) (c)

Figure 3: The bright soliton wave is exhibited through the bell-shaped wave structure of the exponential function solution (29) using the parameter values $b = 0.2$, $c = 0.5$, $\alpha = -0.16$, $\beta = -0.4$, $d = 0.8$, $\omega = 0.2$, and $a = -5$ in the interval $-20 \leq t, x \leq 20$.

2.2.3 Exact solutions of (2) using (G'/G)-expansion approach

One presents the (G'/G)-expansion technique [35] in this subsection to procure analytic solutions of (1+1)D-gnDDE (2).

We observe a solution established as,

$$U(\zeta) = \sum_{j=0}^{M} A_j \left(\frac{E'(\zeta)}{E(\zeta)}\right)^j, \tag{30}$$

where $E(\zeta)$ fulfills,

$$E''(\zeta) + \lambda E'(\zeta) + \mu E(\zeta) = 0, \tag{31}$$

with μ as well as λ taken as constants. Now, M as well as parameters $A_0, \ldots, A_M$, are treated the same way as earlier mentioned.

Engagement of the balancing algorithm to NLNODE (7) gives $M = 2$, consequently, solution of (7) is of the form,

$$U(\zeta) = A_0 + A_1 \left(\frac{E'(\zeta)}{E(\zeta)}\right) + A_2 \left(\frac{E'(\zeta)}{E(\zeta)}\right)^2. \tag{32}$$

Invoking (30) into (7) and engaging (31) procures an algebraic equation in diverse powers of $E(\zeta)$. Following the procedure highlighted previously, we have the system of equations:

$$\begin{aligned}
&c\mu\omega^2 A_1\lambda^3 + d\mu A_1\lambda^3 + 14d\mu^2 A_2\lambda^2 + 14c\mu^2\omega^2 A_2\lambda^2 + 8d\mu^2 A_1\lambda + 8c\mu^2\omega^2 A_1\lambda \\
&+\mu\omega^2 A_1\lambda + a\mu A_1\lambda + b\mu A_0 A_1\lambda + b\mu^2 A_1^2 + 16d\mu^3 A_2 + 2a\mu^2 A_2 + 16c\mu^3\omega^2 A_2 \\
&+2\mu^2\omega^2 A_2 + 2b\mu^2 A_0 A_2 = 0, \\
&c\omega^2 A_1\lambda^4 + dA_1\lambda^4 + 30c\mu\omega^2 A_2\lambda^3 + 30d\mu A_2\lambda^3 + 22c\mu\omega^2 A_1\lambda^2 + \omega^2 A_1\lambda^2 + aA_1\lambda^2 \\
&+22d\mu A_1\lambda^2 + bA_0 A_1\lambda^2 + 3b\mu A_1^2\lambda + 120d\mu^2 A_2\lambda + 120c\mu^2\omega^2 A_2\lambda + 6\mu\omega^2 A_2\lambda \\
&+6a\mu A_2\lambda + 6b\mu A_0 A_2\lambda + 16d\mu^2 A_1 + 16c\mu^2\omega^2 A_1 + 2\mu\omega^2 A_1 + 2a\mu A_1 + 2b\mu A_0 A_1 \\
&+6b\mu^2 A_1 A_2 = 0,
\end{aligned}$$

$$
\begin{aligned}
&16c\omega^2A_2\lambda^4+16dA_2\lambda^4+15c\omega^2A_1\lambda^3+15dA_1\lambda^3+2bA_1^2\lambda^2+232c\mu\omega^2A_2\lambda^2\\
&+4\omega^2A_2\lambda^2+4aA_2\lambda^2+232d\mu A_2\lambda^2+4bA_0A_2\lambda^2+60c\mu\omega^2A_1\lambda+3\omega^2A_1\lambda\\
&+3aA_1\lambda+60d\mu A_1\lambda+3bA_0A_1\lambda+15b\mu A_1A_2\lambda+4b\mu A_1^2+6b\mu^2A_2^2+136d\mu^2A_2\\
&+136c\mu^2\omega^2A_2+8\mu\omega^2A_2+8a\mu A_2+8b\mu A_0A_2, 130c\omega^2A_2\lambda^3+130dA_2\lambda^3\\
&+50c\omega^2A_1\lambda^2+50dA_1\lambda^2+9bA_1A_2\lambda^2+5bA_1^2\lambda+14b\mu A_2^2\lambda+440c\mu\omega^2A_2\lambda\\
&+10\omega^2A_2\lambda+10aA_2\lambda+440d\mu A_2\lambda+10bA_0A_2\lambda+40c\mu\omega^2A_1+2\omega^2A_1\\
&+2aA_1+40d\mu A_1+2bA_0A_1+18b\mu A_1A_2=0,\\
&8bA_2^2\lambda^2+330c\omega^2A_2\lambda^2+330dA_2\lambda^2+60c\omega^2A_1\lambda+60dA_1\lambda+21bA_1A_2\lambda\\
&+3bA_1^2+16b\mu A_2^2+240c\mu\omega^2A_2+6\omega^2A_2+6aA_2+240d\mu A_2+6bA_0A_2=0,\\
&24cA_1\omega^2+336c\lambda A_2\omega^2+18b\lambda A_2^2+24dA_1+336d\lambda A_2+12bA_1A_2=0,\\
&120cA_2\omega^2+10bA_2^2+120dA_2=0.
\end{aligned}
$$

Solving this system of algebraic equations by utilizing the Mathematica software package, one can achieve:

$$
\begin{aligned}
&A_0=-\frac{1}{b}\left\{a+c\lambda^2\omega^2+8c\mu\omega^2+d\lambda^2+8d\mu+\omega^2\right\},\\
&A_1=-\frac{1}{b}\left\{12\lambda\left(c\omega^2+d\right)\right\},\ \ A_2=-\frac{1}{b}\left\{12\left(c\omega^2+d\right)\right\}.
\end{aligned}
$$

Therefore, we can identify the subsequent three categories of traveling wave solutions for (1+1)D-gnDDE (2) in the cases listed as below:

Case 1: When $\lambda^2-4\mu>0$, we achieve the solution in terms of hyperbolic functions,

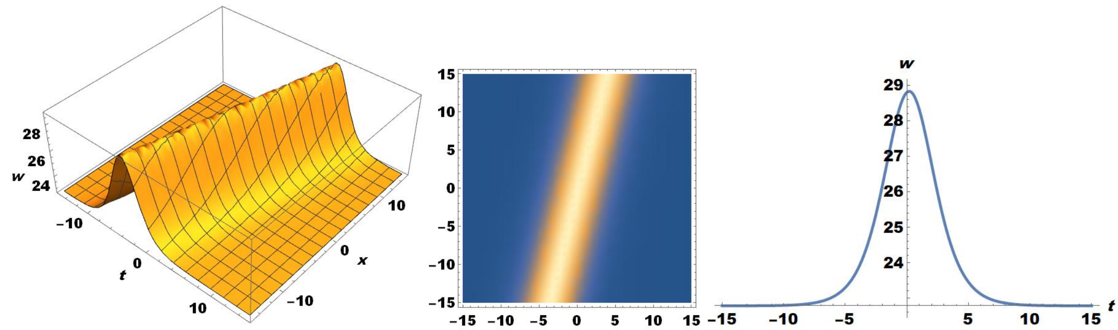
$$
\begin{aligned}
w(x,t)=-\frac{1}{b}\Bigg\{&a+c\lambda^2\omega^2+8c\mu\omega^2+d\lambda^2+8d\mu+\omega^2+\left\{12\lambda\left(c\omega^2+d\right)\right\}\\
&\times\left(\Delta_1\frac{C_0\sinh\left[\Delta_1(x-\omega t)\right]+C_1\cosh\left[\Delta_1(x-\omega t)\right]}{C_0\cosh\left[\Delta_1(x-\omega t)\right]+C_1\sinh\left[\Delta_1(x-\omega t)\right]}-\frac{\lambda}{2}\right)+\left\{12\left(c\omega^2+d\right)\right\}\\
&\times\left(\Delta_1\frac{C_0\sinh\left[\Delta_1(x-\omega t)\right]+C_1\cosh\left[\Delta_1(x-\omega t)\right]}{C_0\cosh\left[\Delta_1(x-\omega t)\right]+C_1\sinh\left[\Delta_1(x-\omega t)\right]}-\frac{\lambda}{2}\right)^2\Bigg\},
\end{aligned}
\tag{33}
$$

where $\Delta_1=\frac{1}{2}\sqrt{\lambda^2-4\mu}$ with constant C_0 as well as C_1 arbitrary. One presents the wave dynamics of the hyperbolic secant-cosecant solution (33) in the plots exhibited in Figure 4.

Case 2: When $\lambda^2-4\mu<0$, we achieve the solution in terms of trigonometric functions,

$$
\begin{aligned}
w(x,t)=-\frac{1}{b}\Bigg\{&a+c\lambda^2\omega^2+8c\mu\omega^2+d\lambda^2+8d\mu+\omega^2+\left\{12\lambda\left(c\omega^2+d\right)\right\}\\
&\times\left(\Delta_2\frac{-C_0\sin\left[\Delta_2(x-\omega t)\right]+C_1\cos\left[\Delta_2(x-\omega t)\right]}{C_0\cos\left[\Delta_2(x-\omega t)\right]+C_1\sin\left[\Delta_2(x-\omega t)\right]}-\frac{\lambda}{2}\right)+\left\{12\left(c\omega^2+d\right)\right\}\\
&\times\left(\Delta_2\frac{-C_0\sin\left[\Delta_2(x-\omega t)\right]+C_1\cos\left[\Delta_2(x-\omega t)\right]}{C_0\cos\left[\Delta_2(x-\omega t)\right]+C_1\sin\left[\Delta_2(x-\omega t)\right]}-\frac{\lambda}{2}\right)^2\Bigg\},
\end{aligned}
\tag{34}
$$

where $\Delta_2=\frac{1}{2}\sqrt{4\mu-\lambda^2}$, with constants C_0 and C_1 arbitrary.



(a) (b) (c)

Figure 4: Bell-shaped wave structure of hyperbolic solution (33) using the data values $b = 0.2$, $c = 0.5$, $\alpha = 0.1$, $\beta = 0.4$, $d = 0.1$, $\omega = 0.2$, $a = -5$, $\lambda = -0.4$, $\mu = -0.8$, $C_0 = 30$, $C_1 = 2$ in the interval $-15 \leq t, x \leq 15$ is displayed in Figure 4. This actually portrays a bright soliton wave structure.

Case 3: When $\lambda^2 - 4\mu = 0$, we achieve the solution in terms of rational function,

$$w(x,t) = -\frac{1}{b}\Bigg\{a + c\lambda^2\omega^2 + 8c\mu\omega^2 + d\lambda^2 + 8d\mu + \omega^2 + \left\{12\lambda\left(c\omega^2 + d\right)\right\} \times \left(\frac{C_1}{C_0 + C_1(x-\omega t)} - \frac{\lambda}{2}\right) + \left\{12\left(c\omega^2 + d\right)\right\}\left(\frac{C_1}{C_0 + C_1(x-\omega t)} - \frac{\lambda}{2}\right)^2\Bigg\}, \tag{35}$$

where constants C_1 and C_2 are arbitrary.

2.3 Significance of the graphical depictions of solutions

We notice that Figure 1 illustrates the periodic wave motion of the elliptic solution (15), Figure 2 (bright-dark combo soliton) reveals the dynamics of the secant hyperbolic solution (21), whereas the bright soliton wave patterns in Figures 3 and 4 display the wave structures of solutions (29) and (33), respectively. These wave depictions are interesting and have notable relevance in the field of science and engineering.

A periodic wave refers to any repeating pattern that maintains a consistent wavelength and frequency (see Figure 5). A periodic wave can be characterized by features like amplitude, frequency, and time period. Examples include sound waves (longitudinal), water waves, light waves (transverse), and alternating current generators. Periodic waves, recognized for their recurring patterns, play an essential role in numerous domains as they underpin radio/audio transmission, AC power, and signal processing, facilitating technologies such as oscilloscopes (see Figure 6) and waveform generators for diagnostic and troubleshooting purposes. The amplitude of a wave, represented as A in the diagram below , is directly linked to the energy of a wave and indicates the wave's highest and lowest points [17].

Bright solitons, together with dark solitons, are localized wave packets that preserve their shape while traveling. They are important in numerous areas such as optics, fluid dynamics, and plasma physics, facilitating effective information transmission and additional applications. Bright solitons represent peaks in intensity, whereas dark solitons signify dips in intensity, with both serving distinct functions in nonlinear wave phenomena.

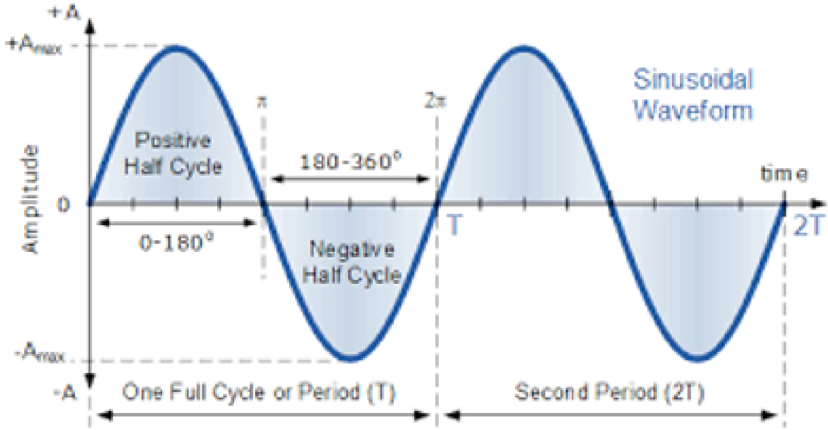


Figure 5: Diagrammatic display of a periodic wave in which the amplitude of the wave, shown as A in the diagram, is directly related to the energy of the wave [17].

The terms "bright" and "dark" originate from optics, where they refer to the luminous areas and shadowy spots that arise in optical fibers . Nonetheless, in the 1830s, soliton sightings were recorded in water. Oceanographers were surprised by the finding in the 1960s and 1970s that luminous solitons existed on the surface of deep ocean waters. Nevertheless, many experiments have been conducted to explore and confirm the phenomenon, with some identifying bright solitons as the source of rogue waves in the ocean. Both bright and dark solitons have currently been detected in Bose-Einstein condensates, plasmas, fiber optics, and various other settings [18].

It is widely recognized that bright soliton shapes are defined by the hyperbolic secant function. The bright soliton solution typically exhibits a bell-shaped form and travels without distortion, maintaining its shape over indefinitely long distances. Nonetheless, dark soliton solutions, arranged as topological optical solitons as well, are represented by the hyperbolic tangent. It is noteworthy that (13) resembles the ordinary differential equation derived in the seminal study by Korteweg and de Vries [19]. This ODE pertains to long waves traveling through a rectangular canal. The ODE (13) characterizes stationary waves, and by applying specific conditions, such as ensuring the fluid remains undisturbed at infinity, Korteweg and de Vries derived both negative and positive solitary wave solutions, along with cnoidal wave solutions [19, 10].

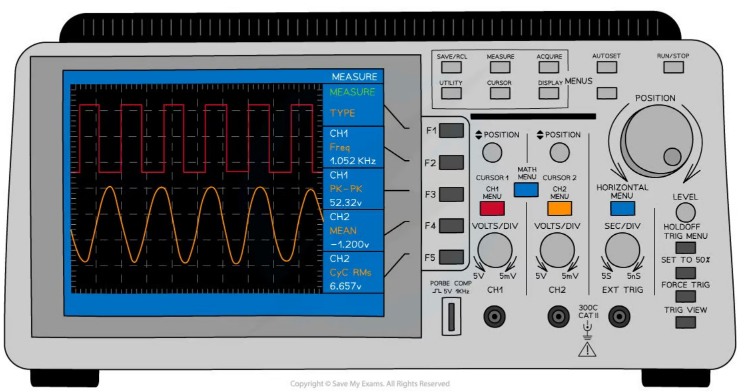


Figure 6: A typical oscilloscope. It contains numerous dials and buttons, yet their primary function is to show and measure fluctuating signals such as sound waves and alternating current. https://www.savemyexams.com/o-level/physics/cie/23/revision-notes/4-electricity-and-magnetism/4-6-uses-of-an-oscilloscope/uses-of-an-oscilloscope/.

3 Conservation Laws of (1+1)D-gnDDE (2)

This segment provides the conserved vectors of the fundamental equation by applying Ibragimov's theorem on preserved vectors, as referenced in previous works [16]. This is achieved by utilizing the established symmetries.

3.1 Formal Lagrangian and conserved currents

Consider a system of sth-order α PDEs [16],

$$\Xi_\sigma(x, \Theta, \Theta_{(1)}, \dots, \Theta_{(s)}) = 0, \quad \sigma = 1, \dots, \alpha, \tag{36}$$

with κ independent together with α dependent variables given as $x = (x^1, x^2, \dots, x^\kappa)$ and $\Theta = (\Theta^1, \Theta^2, \dots, \Theta^\alpha)$. The system of adjoint equations are given by,

$$\Xi^*_\sigma(x, \Theta, \Omega, \dots, \Theta_{(s)}, \Omega_{(s)}) \equiv \frac{\delta(\Omega^\beta \Xi_\beta)}{\delta \Theta^\sigma} = 0, \quad \sigma = 1, \dots, \alpha, \tag{37}$$

where $\Omega = (\Omega^1, \dots, \Omega^\alpha)$ are new dependent variables, $\Omega = \Omega(x)$. The operator $\delta/\delta\Theta^\sigma$, expressed for each σ, as,

$$\frac{\delta}{\delta \Theta^\sigma} = \frac{\partial}{\partial \Theta^\sigma} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\delta}{\delta \Theta^\sigma_{i_1, i_2, \dots, i_s}}, \quad i = 1, \dots, \kappa, \tag{38}$$

is the Euler-Lagrange operator and,

$$D_i = \frac{\partial}{\partial x^i} + \Theta^\sigma_i \frac{\partial}{\partial \Theta^\sigma} + \Theta^\sigma_{ij} \frac{\partial}{\partial \Theta^\sigma_j} + \dots, \quad i = 1, \dots, \kappa, \;\; j = 1, \dots, \kappa, \tag{39}$$

is the total differential operator.

An n-tuple $C = (C^1, C^2, \cdots, C^n)$, such that,

$$D_i C^i = 0, \tag{40}$$

holds for all solutions of (36) is referred to as the conserved current of the equation.

The formal Lagrangian of the system (36) and its adjoint (37) is given as,

$$\mathcal{L} = \Omega^\sigma \Xi_\sigma(x, \Theta, \Theta_{(1)}, \dots, \Theta_{(s)}). \tag{41}$$

Theorem 3.1. *Every nonlocal symmetry, Lie-Bäcklund, as well as Lie point symmetry,*

$$\mathcal{R} = \xi^i \frac{\partial}{\partial x^i} + \varphi^\sigma \frac{\partial}{\partial_{\Theta^\sigma}}, \quad \xi^i = \xi^i(x, \Theta), \quad \varphi^\sigma = \varphi^\sigma(x, \Theta), \tag{42}$$

admitted by the system (36) produces a conserved vector for (36) and its adjoint (37), with the conserved vectors $T = (T^1, \dots, T^\kappa)$ having components T^i given by,

$$\begin{aligned} T^i = \xi^i \mathcal{L} + \Pi^\sigma \left[\frac{\partial \mathcal{L}}{\partial \Theta^\sigma_i} - D_j \frac{\partial \mathcal{L}}{\partial \Theta^\sigma_{ij}} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial \Theta^\sigma_{ijk}} \right) + \dots \right] + D_j(\Pi^\sigma) \left[\frac{\partial \mathcal{L}}{\partial \Theta^\sigma_{ij}} \right. \\ \left. - D_k \frac{\partial \mathcal{L}}{\partial \Theta^\sigma_{ijk}} + \dots \right] + D_j D_k(\Pi^\sigma) \frac{\partial \mathcal{L}}{\partial \Theta_{ijk}} + \dots, \quad i, j, k = 1, \dots, \kappa, \end{aligned} \tag{43}$$

with Lie characteristic function Π^σ *explicated by,*

$$\Pi^\sigma = \varphi^\sigma - \xi^j \Theta^\sigma_j, \quad \sigma = 1, \dots, \alpha, \quad j = 1, \dots, \kappa. \tag{44}$$

Remark 3.1. *It is noted that a set of differential equations (36) is considered self-adjoint when substituting* $v = w$ *into the set of adjoint equations specified in (37) results in the same set of equations (37). For a more thorough understanding of the demonstration and additional valuable information related to the findings discussed here, it is recommended for the reader to refer to the sources cited [16].*

The multiplier Λ of system (36) has the property that,

$$D_i C^i = \Lambda^\sigma \Xi_\sigma, \quad \sigma = 1, \dots, \alpha. \tag{45}$$

The equations governing all multipliers are derived from,

$$\frac{\delta}{\delta \Theta^\sigma} (\Lambda^\sigma \Xi_\sigma) = 0, \quad \sigma = 1, \dots, \alpha. \tag{46}$$

Once the multipliers are produced through (46), the conserved currents can be obtained by using (45) as the governing formula. Now, we proceed to calculate the symmetries of (2) in order to use them for computing the conserved vectors via Theorem 3.1 with the formula (43).

3.2 Conserved vectors of (2) via Ibragimov's theorem

Ibragimov's theorem states that for every conserved quantity in a differential equation, there exists a unique connection to a Lie point symmetry. Therefore, we utilize the previously presented symmetry operators to generate new conserved currents using Ibragimov's theorem [16].

Thus, we give the following theorem:

Theorem 3.2. *If the Euler operator* $\delta/\delta w$ *as explicated in [16] is given consideration, thus, associated adjoint equation of (1+1)D-gnDDE (2) [16] can be expressed through the relation,*

$$H^* \equiv \frac{\delta}{\delta w} \left[v \left\{ w_{tt} + a w_{xx} + b(w w_{xx} + w_x^2) + c w_{ttxx} + d w_{xxxx} \right\} \right] = 0. \tag{47}$$

Further expansion of (47) secures,

$$H^* \equiv v_{tt} + (a + bw) v_{xx} + c v_{ttxx} + d v_{xxxx} = 0. \tag{48}$$

The formal Lagrangian of (1+1)D-gnDDE (2) together with its adjoint presented in (48) is expressed in the format,

$$\mathcal{L} = v \left\{ w_{tt} + a w_{xx} + b(w w_{xx} + w_x^2) + c w_{ttxx} + d w_{xxxx} \right\}. \tag{49}$$

Therefore, the conserved vectors (T^i, X^i), $i = 1, 2, \dots, 6$, *are formulated for the Lagrangian* $(\mathcal{L})$ *by employing the appropriate structure of (43) applicable here, purveyed as [16]:*

$$\begin{aligned} T = \xi^1 \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial w^\alpha_i} - D_j \frac{\partial \mathcal{L}}{\partial w^\alpha_{ij}} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial w^\alpha_{ijk}} \right) + \dots \right] \\ + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial w^\alpha_{ij}} - D_k \frac{\partial \mathcal{L}}{\partial w^\alpha_{ijk}} + \dots \right] + D_j D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial w_{ijk}} + \dots, \end{aligned} \tag{50}$$

$$X = \xi^2\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial w_i^\alpha} - D_j\frac{\partial\mathcal{L}}{\partial w_{ij}^\alpha} + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial w_{ijk}^\alpha}\right) + \dots\right] + D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial w_{ij}^\alpha} - D_k\frac{\partial\mathcal{L}}{\partial w_{ijk}^\alpha} + \dots\right] + D_jD_k(W^\alpha)\frac{\partial\mathcal{L}}{\partial w_{ijk}} + \dots, \tag{51}$$

with constant $\alpha = 1,2$ *as well as* $j = 1,2,3,4$. $W^\alpha = \Psi^\alpha - \xi^j w_j^\alpha$ *is the involved Lie characteristic function.*

One calculates the associated conservation laws of (1+1)D-gnDDE (2) related to vectors Q_1, Q_2 and $Q_1 + \omega Q_2$ using both Theorem 3.2 and the data found in the references cited, as demonstrated by [16]. Therefore, one has:

$$\begin{aligned}
T_1^t &= aw_{xx}v + bw_x^2v + bw_{xx}wv + \frac{1}{2}cvw_{ttxx} + dw_{xxxx}v - \frac{1}{3}cw_{tx}v_{tx} + \frac{1}{6}cv_tw_{txx} \\
&\quad + \frac{1}{2}cw_tv_{txx} - \frac{1}{6}cw_{tt}v_{xx} + \frac{1}{3}cv_xw_{ttx} + w_tv_t, \\
T_1^x &= bw_tv_xw - avw_{tx} - bw_tw_xv - bwvw_{tx} - \frac{1}{2}cvw_{tttx} - dvw_{txxx} + aw_tv_x \\
&\quad + \frac{1}{2}cw_tv_{ttx} - \frac{1}{3}cw_{tt}v_{tx} - \frac{1}{6}cv_{tt}w_{tx} + \frac{1}{3}cv_tw_{ttx} + \frac{1}{6}cw_{ttt}v_x + dw_tv_{xxx} \\
&\quad - dv_{xx}w_{tx} + dv_xw_{txx}, \\
T_2^t &= \frac{1}{6}cv_tw_{xxx} - \frac{1}{2}cvw_{txxx} - vw_{tx} - \frac{1}{6}cv_{xx}w_{tx} - \frac{1}{3}cw_{xx}v_{tx} \\
&\quad + \frac{1}{3}cv_xw_{txx} + \frac{1}{2}cw_xv_{txx} + v_tw_x, \\
T_2^x &= bw_xv_xw + \frac{1}{2}cvw_{ttxx} + w_{tt}v + aw_xv_x + \frac{1}{6}cv_xw_{ttx} - \frac{1}{3}cw_{tx}v_{tx} + \frac{1}{3}cv_tw_{txx} \\
&\quad - \frac{1}{6}cv_{tt}w_{xx} + \frac{1}{2}cw_xv_{ttx} + dw_{xxx}v_x - dw_{xx}v_{xx} + dw_xv_{xxx}, \\
T_3^t &= aw_{xx}v + bw_x^2v + bw_{xx}wv - \frac{1}{2}c\omega vw_{txxx} + \frac{1}{2}cvw_{ttxx} + dw_{xxxx}v \\
&\quad - \omega vw_{tx} + \frac{1}{2}c\omega w_xv_{txx} + \frac{1}{6}c\omega v_tw_{xxx} - \frac{1}{6}c\omega v_{xx}w_{tx} - \frac{1}{3}c\omega w_{xx}v_{tx} + \frac{1}{3}c\omega v_xw_{txx} \\
&\quad - \frac{1}{3}cw_{tx}v_{tx} + \frac{1}{6}cv_tw_{txx} + \frac{1}{2}cw_tv_{txx} - \frac{1}{6}cw_{tt}v_{xx} + \frac{1}{3}cv_xw_{ttx} + \omega v_tw_x + w_tv_t, \\
T_3^x &= bw_tv_xw - avw_{tx} + b\omega w_xv_xw - bw_tw_xv - bwvw_{tx} + \frac{1}{2}c\omega vw_{ttxx} \\
&\quad - \frac{1}{2}cvw_{tttx} - dvw_{txxx} + \omega w_{tt}v + aw_tv_x + a\omega w_xv_x + \frac{1}{6}c\omega v_xw_{ttx} \\
&\quad - \frac{1}{3}c\omega w_{tx}v_{tx} + \frac{1}{3}c\omega v_tw_{txx} - \frac{1}{6}c\omega v_{tt}w_{xx} + \frac{1}{2}c\omega w_xv_{ttx} + \frac{1}{6}cw_{ttt}v_x \\
&\quad - \frac{1}{3}cw_{tt}v_{tx} - \frac{1}{6}cv_{tt}w_{tx} + \frac{1}{3}cv_tw_{ttx} + \frac{1}{2}cw_tv_{ttx} + dv_xw_{txx} + dw_tv_{xxx} \\
&\quad - dv_{xx}w_{tx} + d\omega w_{xxx}v_x - d\omega w_{xx}v_{xx} + d\omega w_xv_{xxx}.
\end{aligned}$$

3.3 Conserved vectors of (2) via the multiplier approach

In this subsection, we utilize the multiplier technique [29] to form conservation laws for the model (1+1)D-gnDDE (2). First of all, we calculate the zeroth-order multiplier

$\mathcal{M}(x,t,w)=\mathcal{M}$ for (2). Thus, following the procedure introduced by [29] and invoking (38), one secures determining equations through the expansion of,

$$\frac{\delta}{\delta w}\left\{\mathcal{M}(t,x,w)\left[w_{tt}+aw_{xx}+b(ww_{xx}+w_x^2)+cw_{ttxx}+dw_{xxxx}\right]\right\}=0, \tag{52}$$

which solves to give the zeroth-order multiplier $\mathcal{M}(t,x,w)$ for (2) as,

$$\mathcal{M}(t,x,w)=(C_1x+C_2)\,t+C_3x+C_4, \tag{53}$$

in which $C_m, m=1,2,3,4$ are arbitrary constants. The associated four multipliers are expressed as,

$$\mathcal{M}_1=xt,\quad \mathcal{M}_2=t,\quad \mathcal{M}_3=x\quad \text{and}\quad \mathcal{M}_4=1. \tag{54}$$

Consequently, relative to the multipliers above, one could through (45) which in this regard expands to,

$$D_xC^x+D_tC^x=\mathcal{M}(t,x,w)\left[w_{tt}+aw_{xx}+b(ww_{xx}+w_x^2)+cw_{ttxx}+dw_{xxxx}\right],$$

calculate the following four conserved vectors of (2) as:

$$\begin{aligned}
C_1^t&=\frac{1}{3}cw_x-w_x-\frac{1}{6}cxw_{xx}+txw_t-\frac{1}{3}ctw_{tx}+\frac{1}{2}ctxw_{txx},\\
C_1^x&=\frac{1}{3}cw_t+btxww_x+atxw_x+dtxw_{xxx}+\frac{1}{2}ctxw_{ttx}-atw-dtw_{xx}-\frac{1}{3}cxw_{tx}\\
&\quad-\frac{1}{6}ctw_{tt}-\frac{1}{2}btw^2,\\
C_2^t&=\frac{1}{2}ctw_{txx}+tw_t-\frac{1}{6}cw_{xx}-w,\\
C_2^x&=\frac{1}{2}ctw_{ttx}+btww_x+atw_x+dtw_{xxx}-\frac{1}{3}cw_{tx},\\
C_3^t&=\frac{1}{2}cxw_{txx}-\frac{1}{3}cw_{tx}+xw_t,\\
C_3^x&=\frac{1}{2}cxw_{ttx}+bxww_x+axw_x+dxw_{xxx}-\frac{1}{2}bw^2-aw-dw_{xx}-\frac{1}{6}cw_{tt},\\
C_4^t&=w_t+\frac{1}{2}cw_{txx},\\
C_4^x&=\frac{1}{2}cw_{ttx}+bww_x+aw_x+dw_{xxx}.
\end{aligned}$$

Remark 3.2. *One notices here, that the multiplier $\mathcal{M}_4(t,x,w)=1$, produces the model (1+1)D-gnDDE (2) in a conserved structure.*

4 Conclusions

The double dispersion equation is important due to its numerous physical applications, such as examining the nonlinear wave distribution in waveguides, investigating the interaction of waveguides with the surrounding medium, and assessing the probability of energy transfer through lateral waveguide coverings. In view of this, this article explores analytical examinations of a $(1+1)$-dimensional generalized double dispersion equation in inhomogeneous and uniform Murnaghan's rod. This is applicable for modeling wave propagation in elastic solid materials, which hold significance in solid-state mechanics. Therefore, the analytical investigations carried out on the generalized double dispersion equation in heterogeneous and uniform Murnaghan's rod expressed in (2) are explicated in this article. Initially, the Lie symmetry analysis approach was used to calculate the Lie point symmetries of the model, resulting in a two-dimensional Lie algebra. Moreover, the obtained nonlinear ordinary differential equation is directly integrated in order to obtain closed-form solutions for the model, achieving an elliptic cosine function solution in this case. By employing a specific conversion, the method also guarantees the achievement of a Weierstrass function solution.

Furthermore, to find more answers to the given problem, the popular Kudryashov's technique was employed, allowing us to obtain a solution in exponential form. Following that, we utilized the (G'/G)-expansion method, which in turn generates solutions in the form of hyperbolic, trigonometric, and rational functions. Furthermore, showcasing the wave dynamics of the obtained solutions helps us better understand the physical interpretations of these solutions, with various representations including 3-dimensional, 2-dimensional, contour, and density plots. In conclusion, the research also identifies important conserved quantities like energy, mass, and momentum, which are upheld by utilizing Ibragimov's theorem and the multiplier method. The two techniques applied ensure that various conservation laws of note are derived, including the conservation of momentum and energy.

Therefore, the findings may prove beneficial to various researchers within the realms of science and engineering. We add that despite the fact that the main method utilized here can only be applied to differential equations with symmetries, it is one of the best approaches to solving differential equations. Additionally, the work does not cover the use of conservation laws of the studied model to obtain solutions; therefore, future work could involve the application of the associated conserved vectors to perform multiple reductions to secure more analytical solutions to the model.

Acknowledgement The authors express their gratitude to the Mafikeng campus of North-West University for their assistance.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] M. Abramowitz & I. A. Stegun (1965). *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*. Courier Corporation, Washington D. C.

[2] O. D. Adeyemo, C. M. Khalique, Y. S. Gasimov & F. Villecco (2023). Variational and non-variational approaches with lie algebra of a generalized (3+1)-dimensional nonlinear potential Yu-Toda-Sasa-Fukuyama equation in Engineering and Physics. *Alexandria Engineering Journal*, *63*, 17–43. https://doi.org/10.1016/j.aej.2022.07.024.

[3] O. D. Adeyemo, L. Zhang & C. M. Khalique (2022). Optimal solutions of lie subalgebra, dynamical system, travelling wave solutions and conserved currents of (3+1)-dimensional generalized Zakharov–Kuznetsov equation type I. *The European Physical Journal Plus*, *137*(8), Article ID: 954. https://doi.org/10.1140/epjp/s13360-022-03100-z.

[4] N. I. Akhiezer (1990). *Elements of The Theory of Elliptic Functions*. American Mathematical Society, Providence, Rhode Island.

[5] C. Cattani, T. A. Sulaiman, H. M. Baskonus & H. Bulut (2018). Solitons in an inhomogeneous Murnaghan's rod. *The European Physical Journal Plus*, *133*, 1–11. https://doi.org/10.1140/epjp/i2018-12085-y.

[6] Y. Chen & Z. Yan (2005). New exact solutions of (2+1)-dimensional Gardner equation via the new sine-Gordon equation expansion method. *Chaos, Solitons & Fractals*, *26*(2), 399–406. https://doi.org/10.1016/j.chaos.2005.01.004.

[7] C. Chun & R. Sakthivel (2010). Homotopy perturbation technique for solving two-point boundary value problems–comparison with other methods. *Computer Physics Communications*, *181*(6), 1021–1024. https://doi.org/10.1016/j.cpc.2010.02.007.

[8] P. A. Clarkson (1985). The Painlevé property and a partial differential equation with an essential singularity. *Physics Letters A*, *109*(5), 205–208. https://doi.org/10.1016/0375-9601(85)90303-2.

[9] M. T. Darvishi & M. Najafi (2011). A modification of extended homoclinic test approach to solve the (3+1)-dimensional potential-YTSF equation. *Chinese Physics Letters*, *28*(4), Article ID: 040202. https://dx.doi.org/10.1088/0256-307X/28/4/040202.

[10] P. G. Drazin & R. S. Johnson (1989). *Soliton: An Introduction*. Cambridge University Press, New York, NY.

[11] X. X. Du, B. Tian, Q. X. Qu, Y. Q. Yuan & X. H. Zhao (2020). Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov-Kuznetsov equation in an electron-positron-ion magnetoplasma. *Chaos, Solitons & Fractals*, *134*, Article ID: 109709. https://doi.org/10.1016/j.chaos.2020.109709.

[12] L. L. Feng, S. F. Tian, T. T. Zhang & J. Zhou (2017). Lie symmetries, conservation laws and analytical solutions for two-component integrable equations. *Chinese Journal of Physics*, *55*(3), 996–1010. https://doi.org/10.1016/j.cjph.2017.03.008.

[13] X. Y. Gao, Y. J. Guo & W. R. Shan (2020). Water-wave symbolic computation for the Earth, Enceladus and Titan: The higher-order Boussinesq-Burgers system, auto-and non-auto-bäcklund transformations. *Applied Mathematics Letters*, *104*, Article ID: 106170. https://10.1016/j.aml.2019.106170.

[14] I. S. Gradshteyn & I. M. Ryzhik (2014). *Table of Integrals, Series, and Products*. Academic Press, California, USA. https://doi.org/10.1016/C2010-0-64839-5.

[15] C. Gu (2013). *Soliton Theory and Its Applications*. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-03102-5.

[16] N. H. Ibragimov (2007). A new conservation theorem. *Journal of Mathematical Analysis and Applications*, *333*(1), 311–328. https://doi.org/10.1016/j.jmaa.2006.10.078.

[17] iLearn Engineering. What are Periodic Waves? https://www.ilearnengineering.com/electronical-and-electronic/what-are-periodic-waves 2022.

[18] M. A. B. Jalil (2023). The study on the fundamental of dark and bright solitons. *International Journal for Research in Applied Science and Engineering Technology*, *11*(12), 1777–1783. https://doi.org/10.22214/ijraset.2023.57721.

[19] D. J. Korteweg & G. De Vries (1895). XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *39*(240), 422–443. https://doi.org/10.1080/14786449508620739.

[20] N. A. Kudryashov (2004). *Analytical Theory of Nonlinear Differential Equations*. Moskow-Igevsk, Institute of Computer Investigations, Moscow, Russia.

[21] N. A. Kudryashov & N. B. Loguinova (2008). Extended simplest equation method for nonlinear differential equations. *Applied Mathematics and Computation*, *205*(1), 396–402. https://doi.org/10.1016/j.amc.2008.08.019.

[22] N. A. Kudryashov (2012). One method for finding exact solutions of nonlinear differential equations. *Communications in Nonlinear Science and Numerical Simulation*, *17*(6), 2248–2253. https://doi.org/10.1016/j.cnsns.2011.10.016.

[23] N. A. Kudryashov (2019). First integrals and general solution of the Fokas–Lenells equation. *Optik*, *195*, Article ID: 163135. https://doi.org/10.1016/j.ijleo.2019.163135.

[24] S. Kumar & S. K. Dhiman (2024). Exploring cone-shaped solitons, breather, and lump-forms solutions using the lie symmetry method and unified approach to a coupled breaking soliton model. *Physica Scripta*, *99*(2), Article ID: 025243. https://doi.org/10.1088/1402-4896/ad1d9e.

[25] C. K. Kuo & W. X. Ma (2022). An effective approach for constructing novel KP-like equations. *Waves in Random and Complex Media*, *32*(2), 629–640. https://doi.org/10.1080/17455030.2020.1792580.

[26] L. Li, C. Duan & F. Yu (2019). An improved Hirota bilinear method and new application for a nonlocal integrable complex modified Korteweg-de Vries (MKdV) equation. *Physics Letters A*, *383*(14), 1578–1582. https://doi.org/10.1016/j.physleta.2019.02.031.

[27] D. N. Madhavan, I. Alias, G. Ibragimov & R. M. Hasim (2024). Some results on pursuit games for an infinite system of ternary differential equations. *Malaysian Journal of Mathematical Sciences*, *18*(3), 567–581. https://doi.org/10.47836/mjms.18.3.07.

[28] K. S. Nisar & R. Silambarasan (2024). Classification of jacobi solutions of double dispersion equation in uniform and inhomogeneous Murnaghan's rod. *Partial Differential Equations in Applied Mathematics*, *9*, Article ID: 100624. https://doi.org/10.1016/j.padiff.2024.100624.

[29] P. J. Olver (1993). *Applications of Lie Groups To Differential Equations*. Springer Science & Business Media, New York, NY. https://doi.org/10.1007/978-1-4684-0274-2.

[30] L. V. Ovsiannikov (2014). *Group Analysis of Differential Equations*. Academic Press, New York, NY. https://doi.org/10.1016/C2013-0-07470-1.

[31] A. H. Salas & C. A. Gómez S (2010). Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation. *Mathematical Problems in Engineering*, *2010*(1), 194329. https://doi.org/10.1155/2010/194329.

[32] A. M. Samsonov (2001). *Strain Solitons In Solids and How To Construct Them*. Chapman & Hall/CRC, Boca Raton, Florida. https://doi.org/10.1201/9781420026139.

[33] N. M. L. K. Shamshu, A. Alias & K. C. Loy (2025). Numerical solution of extended Korteweg-de Vries equation with cubic nonlinear term and fifth-order dispersion term. *Malaysian Journal of Mathematical Sciences*, *19*, 163–176. https://doi.org/10.47836/mjms.19.1.09.

[34] N. K. Vitanov (2010). Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity. *Communications in Nonlinear Science and Numerical Simulation*, *15*(8), 2050–2060. https://doi.org/10.1016/j.cnsns.2009.08.011.

[35] M. Wang, X. Li & J. Zhang (2008). The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*, *372*(4), 417–423. https://doi.org/10.1016/j.physleta.2007.07.051.

[36] M. Wang, Y. Zhou & Z. Li (1996). Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Physics Letters A*, *216*(1-5), 67–75. https://doi.org/10.1016/0375-9601(96)00283-6.

[37] A. M. Wazwaz (2002). *Partial Differential Equations*. A. A. Balkema Publishers, Lisse, Netherlands.

[38] A. M. Wazwaz (2007). Travelling wave solutions to (2+1)-dimensional nonlinear evolution equations. *Journal of Natural Sciences and Mathematics*, *1*, 1–13.

[39] X. Zeng & D. S. Wang (2009). A generalized extended rational expansion method and its application to (1+1)-dimensional dispersive long wave equation. *Applied Mathematics and Computation*, *212*(2), 296–304. https://doi.org/10.1016/j.amc.2009.02.020.

[40] C. R. Zhang, B. Tian, Q. X. Qu, L. Liu & H. Y. Tian (2020). Vector bright solitons and their interactions of the couple Fokas–Lenells system in a birefringent optical fiber. *Zeitschrift für Angewandte Mathematik und Physik*, *71*, 1–19. https://doi.org/10.1007/s00033-019-1225-9.

[41] L. Zhang & C. M. Khalique (2018). Classification and bifurcation of a class of second-order ODEs and its application to nonlinear PDEs. *Discrete & Continuous Dynamical Systems-Series S*, *11*(4), 759–772. https://doi.org/10.3934/dcdss.2018048.

[42] Y. Zhang, R. Ye & W. Ma (2020). Binary darboux transformation and soliton solutions for the coupled complex modified Korteweg-de Vries equations. *Mathematical Methods in The Applied Sciences*, *43*(2), 613–627. https://doi.org/10.1002/mma.5914.

[43] Y. Zhou, M. Wang & Y. Wang (2003). Periodic wave solutions to a coupled KdV equations with variable coefficients. *Physics Letters A*, *308*(1), 31–36. https://doi.org/10.1016/S0375-9601(02)01775-9.